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A POLYOMINO WITH NO STOCHASTIC FUNCTION

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Let S denote the set of unit squares with integer vertices in $\mathbf{R}^2$. A *polyomino* is a finite subset of S. If R is a *rectangle* (i.e. a polyomino which is also a rectangle) contained in the polyomino P then R is a *rectangle of* P; R is *maximal* if it is not properly contained in another rectangle of P. A *stochastic function* on P is a function f from P to the nonnegative reals, such that for every maximal rectangle R of P, $\sum_{s\in R} f(s)=1$. Berge et al. gave a sufficient condition (called *pataconvexity*) for a polyomino to admit a stochastic function, and asked whether there is a polyomino which admits no stochastic function. In this note we give an example of such a polyomino P, shown in Figure 1.

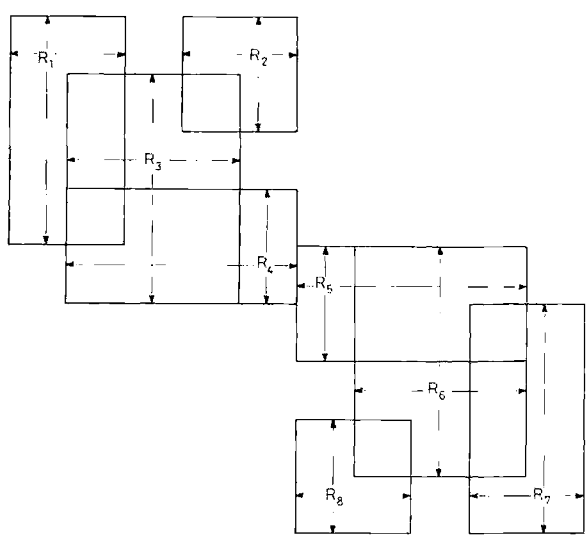

Fig. 1

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Suppose f were a stochastic function for P. Eight maximal rectangles $R_1, \dots, R_8$ are shown in Figure 1 and nine maximal rectangles $Q_1, \dots, Q_9$ are shown in Figure 2. Observe that for any square s the number of R_i containing s is greater than or equal the number of Q_j containing s. Thus

$$\sum_{s \in P} |\{R_i\colon\ s \in R_i\}|\, f(s) \geqq \sum_{s \in P} |\{Q_j\colon\ s \in Q_j\}|\, f(s).$$

But this is impossible, since

$$\sum_{s \in P} |\{R_i\colon\ s \in R_i\}|\, f(s) = \sum_{i=1}^{8} \Big(\sum_{s \in R_i} f(s) \Big) = 8$$

and

$$\sum_{s \in P} |\{Q_j\colon\ s \in Q_j\}|\, f(s) = \sum_{j=1}^{9} \Big(\sum_{s \in Q_j} f(s) \Big) = 9.$$

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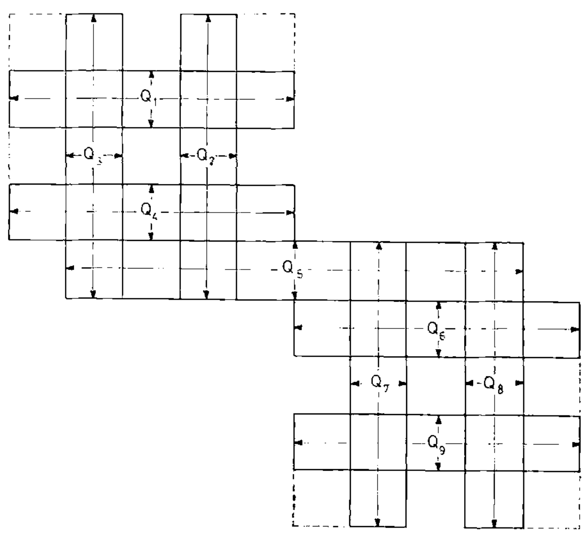

Fig. 2

Reference

[1] C. Berge, C. C. Chen, V. Chvátal and C. S. Seow, Combinatorial properties of polyominoes, *Combinatorica* **1** (1981), 217—224.

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